Chapter 7

Angular Momentum

In this Chapter, we study angular momentum, which is a concept of fundamental importance in mechanics. When we have introduced the momentum operator as well as the Hamiltonian operator, we have made the important observation that these operators emerge as generators of certain physical operations. This is the case of the translations (for the momentum) and of time evolution (for the Hamiltonian). Another fundamental operation that we can perform on a physical system is the ensemble of rotations. We will see in this Chapter that the very concept of angular momentum in quantum mechanics emerges from the properties of rotations. In this Chapter, we first introduce the angular momentum and its commutation relations, and then study the general properties of its spectrum and eigenstates.

7.1 Rotation matrices

We start by recalling some important properties of rotation matrices, that will be instrumental in deriving angular momentum in the quantum setting. Let us consider a 3-dimensional system, specified by a vector of coordinates $\mathbf{v} = (v_x, v_y, v_z) \equiv (v_1, v_2, v_3)$. In general, any rotation can be expressed as the action of a 3×3 matrix $\hat{R}$, such that the transformed coordinates read:

$$\mathbf{v}' = \hat{R}\mathbf{v}. \tag{7.1.1}$$

We can find conditions on the matrix $\hat{R}$ noticing that the scalar product between two rotated vectors must be preserved by the rotation:

$$\mathbf{v}' \cdot \mathbf{w}' = \mathbf{v} \cdot \mathbf{w}, \tag{7.1.2}$$

thus

$$\begin{aligned}
\mathbf{v}' \cdot \mathbf{w}' &= \sum_i v_i' w_i' \\
&= \sum_i \Big(\sum_j R_{ij} v_j \Big) \Big(\sum_k R_{ik} w_k \Big) \\
&= \sum_{j,k} (v_j w_k) \sum_i R_{ij} R_{ik} \\
&= \sum_i v_i w_i \ = \mathbf{v} \cdot \mathbf{w} \,,
\end{aligned} \tag{7.1.3}$$

which is satisfied if the matrix is an orthogonal matrix:

$$\sum_i R_{ij} R_{ik} = \delta_{jk}, \quad \text{or} \quad \hat{R}\hat{R}^T = \hat{I} \,. \tag{7.1.4}$$

In general, all orthogonal matrices have $\det \hat{R} = \pm 1$, but in the following, we will study only rotations such that $\det \hat{R} = 1$. These are called *orientation-preserving rotations*, since they do not involve flipping axes, for example, but rather only continuously changing the coordinate system.

In order to understand the general properties of $\hat{R}$ it is very useful to expand the matrix $\hat{R}$ for small rotations:

$$\hat{R} = \hat{I} + \delta\hat{\rho}, \tag{7.1.5}$$

where $\hat{\rho}$ is a 3×3 matrix we wish to determine and that characterizes both the rotation direction and angle of rotation. We can compute the inverse infinitesimal rotation to be just $\hat{R}^{-1} = \hat{I} - \delta\hat{\rho}$, as it can be easily checked

$$\hat{R}\hat{R}^{-1} = (\hat{I} + \delta\hat{\rho})(\hat{I} - \delta\hat{\rho}) = \hat{I} + \mathcal{O}(\delta^2). \tag{7.1.6}$$

Since the rotation matrix is orthogonal: $\hat{R}\hat{R}^T = \hat{I}$, or equivalently $\hat{R}^T = \hat{R}^{-1}$, we then have

$$\hat{R}^T = \hat{I} - \delta\hat{\rho}, \tag{7.1.7}$$

$$\hat{\rho}^T = -\hat{\rho}, \tag{7.1.8}$$

thus the infinitesimal rotation matrix must be antisymmetric $\rho_{ij} = -\rho_{ji}$, or explicitly written:

$$\hat{\rho} \doteq \begin{pmatrix} 0 & \rho_{12} & \rho_{13} \\ -\rho_{12} & 0 & \rho_{23} \\ -\rho_{13} & -\rho_{23} & 0 \end{pmatrix} . \tag{7.1.9}$$

The three independent components of the matrix $\hat{\rho}$ then fully determine the infinitesimal action of the rotation operator. We can now conveniently define a vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ in terms of the 3 independent components:

$$\theta_1 \equiv -\rho_{23}, \quad \theta_2 \equiv \rho_{13}, \quad \theta_3 \equiv -\rho_{12}. \tag{7.1.10}$$

Notice that the sign convention relating the components of $\boldsymbol{\theta}$ to the matrix elements of $\hat{\rho}$ is arbitrary and boils down to choosing whether the rotations are to be taken clockwise or anti-clockwise. We have used the widely adopted anti-clockwise choice that allows us to write the matrix elements compactly as:

$$\rho_{ij} = -\epsilon_{ijk}\theta_k, \tag{7.1.11}$$

where we have introduced the antisymmetric Levi-Civita symbol

$$\epsilon_{ijk} = \begin{cases} +1 & \text{cyclic permutations of } (1,\, 2,\, 3), \\ -1 & \text{cyclic permutations of } (1,\, 3,\, 2), \\ 0 & i = j,\, j = k \text{ or } i = k. \end{cases} \tag{7.1.12}$$

With this convention, an infinitesimal rotation then reads

$$R_{ij} = \delta_{ij} - \sum_k \epsilon_{ijk}\delta\theta_k + \mathcal{O}(\delta^2), \tag{7.1.13}$$

and the transformed vector

$$v_i' = \sum_j R_{ij}v_j = v_i - \sum_{jk} \epsilon_{ijk}v_j\delta\theta_k. \tag{7.1.14}$$

but the last term is nothing but a cross product, since in general

$$(\mathbf{a} \times \mathbf{b})_k = \sum_{ij} \epsilon_{ijk}a_ib_j, \tag{7.1.15}$$

$$(\mathbf{a} \times \mathbf{b})_i = -\sum_{jk} \epsilon_{ijk}b_ja_k. \tag{7.1.16}$$

Vector-wise we can write the infinitesimal rotation as

$$\mathbf{v}' = \mathbf{v} + (\delta\boldsymbol{\theta} \times \mathbf{v}), \tag{7.1.17}$$

thus the meaning of the vector $\delta\boldsymbol{\theta}$ is really what we would expect from a rotation along the $\boldsymbol{\theta}$ direction, and the vector is rotated by an infinitesimal angle $|\delta\boldsymbol{\theta}|$ along that direction.

7.1.1 Rotations do not commute

A very important feature of rotations is that rotations across different directions do not commute.
In order to explicitly compute the commutator, we can consider two infinitesimal rotations through the directions $\delta\boldsymbol{\alpha}$ and $\delta\boldsymbol{\beta}$. If we apply first the $\boldsymbol{\alpha}$ rotation and then the $\boldsymbol{\beta}$ rotation, we get

$$\hat{R}(\delta\boldsymbol{\beta})\hat{R}(\delta\boldsymbol{\alpha})\mathbf{v} = \hat{R}(\delta\boldsymbol{\beta})\Big(\mathbf{v} + \delta\boldsymbol{\alpha} \times \mathbf{v} + \frac{\delta^2}{2}\hat{S}(\boldsymbol{\alpha})\mathbf{v} + \dots\Big), \tag{7.1.18}$$

$$= \mathbf{v} + \delta\boldsymbol{\alpha} \times \mathbf{v} + \delta\boldsymbol{\beta} \times \mathbf{v} + \delta\boldsymbol{\beta} \times (\delta\boldsymbol{\alpha} \times \mathbf{v}) + \frac{\delta^2}{2}\Big(\hat{S}(\boldsymbol{\alpha})\mathbf{v} + \hat{S}(\boldsymbol{\beta})\mathbf{v}\Big) + \mathcal{O}(\delta^3), \quad (7.1.19)$$

where we have formally introduced the second-order development of the rotation matrix $\hat{S}(\boldsymbol{\theta})$, that we won't compute explicitly for the moment. The composition of the two rotations in the other order gives the same expression but with $\boldsymbol{\alpha} \leftrightarrow \boldsymbol{\beta}$, i.e.,

$$\hat{R}(\delta\boldsymbol{\alpha})\hat{R}(\delta\boldsymbol{\beta})\mathbf{v} = \mathbf{v} + \delta\boldsymbol{\beta} \times \mathbf{v} + \delta\boldsymbol{\alpha} \times \mathbf{v} + \delta\boldsymbol{\alpha} \times (\delta\boldsymbol{\beta} \times \mathbf{v}) + \frac{\delta^2}{2}\Big(\hat{S}(\boldsymbol{\alpha})\mathbf{v} + \hat{S}(\boldsymbol{\beta})\mathbf{v}\Big) + \mathcal{O}(\delta^3). \quad (7.1.20)$$

So we see that the commutator applied on the vector $\mathbf{v}$ is given by

$$\Big[\hat{R}(\delta\boldsymbol{\beta}), \hat{R}(\delta\boldsymbol{\alpha})\Big]\mathbf{v} = \delta\boldsymbol{\beta} \times (\delta\boldsymbol{\alpha} \times \mathbf{v}) - \delta\boldsymbol{\alpha} \times (\delta\boldsymbol{\beta} \times \mathbf{v}) + \mathcal{O}(\delta^3). \quad (7.1.21)$$

To further evaluate this expression we recall that the cross product is not associative but it rather satisfies the Jacobi identity:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0, \quad (7.1.22)$$

and also the elementary antisymmetric property

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}. \quad (7.1.23)$$

Therefore,

$$\begin{aligned} \delta\boldsymbol{\beta} \times (\delta\boldsymbol{\alpha} \times \mathbf{v}) - \delta\boldsymbol{\alpha} \times (\delta\boldsymbol{\beta} \times \mathbf{v}) &= \delta\boldsymbol{\beta} \times (\delta\boldsymbol{\alpha} \times \mathbf{v}) + \delta\boldsymbol{\alpha} \times (\mathbf{v} \times \delta\boldsymbol{\beta}), && (7.1.24) \\ &= -\mathbf{v} \times (\delta\boldsymbol{\beta} \times \delta\boldsymbol{\alpha}), && (7.1.25) \\ &= (\delta\boldsymbol{\beta} \times \delta\boldsymbol{\alpha}) \times \mathbf{v}. && (7.1.26) \end{aligned}$$

The commutator among the infinitesimal rotations then gives an interesting result:

$$\Big[\hat{R}(\delta\boldsymbol{\beta}), \hat{R}(\delta\boldsymbol{\alpha})\Big]\mathbf{v} = (\delta\boldsymbol{\beta} \times \delta\boldsymbol{\alpha}) \times \mathbf{v} = \hat{R}(\delta\boldsymbol{\beta} \times \delta\boldsymbol{\alpha})\mathbf{v} - \mathbf{v}, \quad (7.1.27)$$

and is therefore equivalent to a rotation of an angle $\mathcal{O}(\delta^2)$ in the direction $\boldsymbol{\beta} \times \boldsymbol{\alpha}$. We also see, instead, that rotations in the same direction ($\delta\boldsymbol{\beta}$ parallel to $\delta\boldsymbol{\alpha}$) clearly commute, which means that a non-infinitesimal rotation along a fixed direction can be found by many (N) repeated applications of an infinitesimal rotation of size $\delta = 1/N$:

$$\begin{aligned} \hat{R}(\boldsymbol{\theta}) &= \lim_{N\to\infty} \hat{R}(\delta\boldsymbol{\theta}) \dots \hat{R}(\delta\boldsymbol{\theta}), && (7.1.28) \\ &= \lim_{N\to\infty} \prod_{i=1}^{N} \hat{R}(\boldsymbol{\theta}/N), && (7.1.29) \\ &= \lim_{N\to\infty} \prod_{i=1}^{N} \Big(\hat{I} + \hat{\rho}/N\Big), && (7.1.30) \\ &= \exp(\hat{\rho}). && (7.1.31) \end{aligned}$$

thus an exponential of the 3×3 matrix $\hat{\rho}$. For example, if $\boldsymbol{\theta} = (0, 0, \theta_3)$ one can compute the rotation matrix explicitly as

$$\hat{R}(\theta_3) \doteq \exp \theta_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.1.32}$$

which exactly coincides with the familiar expression for rotations in the z direction. Similarly for the other directions:

$$\hat{R}(\theta_1) \doteq \exp \theta_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & -\sin(\theta_1) \\ 0 & \sin(\theta_1) & \cos(\theta_1) \end{pmatrix}, \tag{7.1.33}$$

which coincides with rotations in the x direction of an angle θ_1, and finally

$$\hat{R}(\theta_2) \doteq \exp \theta_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & 1 & 0 \\ -\sin(\theta_2) & 0 & \cos(\theta_2) \end{pmatrix}, \tag{7.1.34}$$

coinciding with rotations along the y direction. It should be remarked here that we have derived these matrices using purely algebraic techniques and only making use of the elementary property of rotations.

7.2 The Rotation Operator

In quantum mechanics, we assume that every rotation described by $\hat{R}$ is in one-to-one correspondence to some operator that transforms quantum states accordingly, $\hat{D}(\hat{R})$. Since we have seen that rotation matrices are themselves uniquely identified by the vector $\boldsymbol{\theta}$, we will just call the quantum mechanical operator $\hat{D}(\boldsymbol{\theta})$. As we have done for the momentum operator and for the time evolution operator, we can then also associate a quantum mechanical rotation operator to the rotations, such that a rotated ket is written

$$|\psi_{\boldsymbol{\theta}}\rangle = \hat{D}(\boldsymbol{\theta}) |\psi\rangle . \tag{7.2.1}$$

It should be noticed that, while defined in terms of a 3×3 matrix, the operator $\hat{D}(\boldsymbol{\theta})$ however acts on the Hilbert space spanned by state vectors and not, in general, on the regular three-dimensional Cartesian space. The action of this rotation operator therefore needs to be specified in terms of the requested properties on the kets.
As much as done for the other transformations we have studied so far, also for rotations we expect:

$$\langle \psi_{\boldsymbol{\theta}} | \psi_{\boldsymbol{\theta}} \rangle = \left\langle \Psi \middle| \hat{D}^\dagger(\boldsymbol{\theta}) \hat{D}(\boldsymbol{\theta}) |\Psi \right\rangle = \langle \Psi | \Psi \rangle . \tag{7.2.2}$$

Thus the operator is unitary:

$$\hat{D}^\dagger(\boldsymbol{\theta}) \hat{D}(\boldsymbol{\theta}) = \hat{I}. \tag{7.2.3}$$

The second property we expect from this operator is that it can be arbitrarily composed:

$$\hat{D}(\boldsymbol{\theta}_1)\hat{D}(\boldsymbol{\theta}_2) = \hat{D}(\boldsymbol{\theta}_1\boldsymbol{\theta}_2), \tag{7.2.4}$$

where $\boldsymbol{\theta}_1\boldsymbol{\theta}_2$ denotes the composite rotation $\hat{R}(\boldsymbol{\theta}_1\boldsymbol{\theta}_2) \equiv \hat{R}(\boldsymbol{\theta}_1)\hat{R}(\boldsymbol{\theta}_2)$. Furthermore, if we rotate a certain system *back* to its original state, this operation should be equivalent to applying the inverse transformation:

$$\hat{D}(-\boldsymbol{\theta}) = \hat{D}^{-1}(\boldsymbol{\theta}), \tag{7.2.5}$$

where $\hat{D}^{-1}$ denotes the inverse of the operator.
The last property that we can intuitively expect is that in the limit of vanishing rotations the operator $\hat{D}$ should strictly reduce to the identity:

$$\lim_{\delta\to 0} \hat{D}(\delta\boldsymbol{\theta}) = \hat{I}. \tag{7.2.6}$$

As we have already seen for the case of the time evolution operator, and as a consequence of Stone's theorem, all these conditions are satisfied if we take an infinitesimal rotation operator to be described by an exponential of a Hermitian operator:

$$\hat{D}(\delta\boldsymbol{\theta}) = e^{-i\frac{\hat{\mathbf{J}}\cdot\delta\boldsymbol{\theta}}{\hbar}}, \tag{7.2.7}$$

where once again this transformation *defines* the angular momentum operators

$$\hat{\mathbf{J}} \equiv (\hat{J}_1, \hat{J}_2, \hat{J}_3) \equiv (\hat{J}_x, \hat{J}_y, \hat{J}_z). \tag{7.2.8}$$

7.3 Commutation Relations of Angular Momentum Operators

In order to establish the commutation relations among the different components of the angular momentum, we recall that for infinitesimal rotation matrices we have found

$$[\hat{R}(\delta\boldsymbol{\beta}), \hat{R}(\delta\boldsymbol{\alpha})] = \hat{R}(\delta\boldsymbol{\beta}\times\delta\boldsymbol{\alpha}) - \hat{I}. \tag{7.3.1}$$

Since we have postulated a one-to-one correspondence between rotation matrices and rotation operators, this also implies that for infinitesimal rotation operators we must have:

$$[\hat{D}(\delta\boldsymbol{\beta}), \hat{D}(\delta\boldsymbol{\alpha})] = \hat{D}(\delta\boldsymbol{\beta}\times\delta\boldsymbol{\alpha}) - \hat{I}. \tag{7.3.2}$$

The left-hand side of this equation is

$$[\hat{D}(\delta\boldsymbol{\beta}), \hat{D}(\delta\boldsymbol{\alpha})] = \Big(\hat{I} - \frac{i\delta}{\hbar}\hat{\mathbf{J}}\cdot\boldsymbol{\beta}\Big)\Big(\hat{I} - \frac{i\delta}{\hbar}\hat{\mathbf{J}}\cdot\boldsymbol{\alpha}\Big) \tag{7.3.3}$$

$$- \Big(\hat{I} - \frac{i\delta}{\hbar}\hat{\mathbf{J}}\cdot\boldsymbol{\alpha}\Big)\Big(\hat{I} - \frac{i\delta}{\hbar}\hat{\mathbf{J}}\cdot\boldsymbol{\beta}\Big) \tag{7.3.4}$$

$$= -\frac{\delta^2}{\hbar^2}\big[\hat{\mathbf{J}}\cdot\boldsymbol{\beta}, \hat{\mathbf{J}}\cdot\boldsymbol{\alpha}\big]. \tag{7.3.5}$$

and the right-hand side is

$$\hat{D}(\delta\boldsymbol{\beta} \times \delta\boldsymbol{\alpha}) - \hat{I} = -\frac{i}{\hbar}\delta^2 \hat{\mathbf{J}} \cdot (\boldsymbol{\beta} \times \boldsymbol{\alpha}). \tag{7.3.6}$$

Thus, equating the two:

$$\left[\hat{\mathbf{J}} \cdot \boldsymbol{\beta}, \hat{\mathbf{J}} \cdot \boldsymbol{\alpha}\right] = i\hbar \hat{\mathbf{J}} \cdot (\boldsymbol{\beta} \times \boldsymbol{\alpha}). \tag{7.3.7}$$

This equation must hold for arbitrary directions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, thus we can take for example them to be parallel to two of the unit vectors $\boldsymbol{\alpha} = \alpha_j \mathbf{e}_j$ and $\boldsymbol{\beta} = \beta_i \mathbf{e}_i$, yielding

$$[\hat{J}_i, \hat{J}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{J}_k. \tag{7.3.8}$$

This relation is most often written dropping the explicit summation over k, since that is redundant, and it yields the following compact expression for the commutators of the different components of the angular momentum:

$$[\hat{J}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{J}_k. \tag{7.3.9}$$

The latter result is the fundamental commutator relation for angular momentum operators and, together with the canonical commutation relations, is one of the cornerstones of quantum theory.
A remarkable difference with respect to the case of linear momentum, $\hat{\mathbf{P}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$, is that in this case the generators of the angular momentum do not commute among themselves. The lack of commutativity of the rotation matrices $\hat{R}$ then directly implies also the lack of commutation of the $\hat{\mathbf{J}}$ operators.

7.3.1 Finite Rotations

We have analyzed so far the rotation operator for infinitesimal transformations, of order δ. As a consequence of the non-commutativity of the angular momentum operators in the different directions, also rotations along different directions do not commute, and the order in which they are realized is important. However, rotations along the same direction clearly commute, thus we can write the finite rotation operator as a product of many infinitesimal transformations, in such a way that for finite rotations:

$$\hat{D}(\boldsymbol{\theta}) = e^{-i\frac{\hat{\mathbf{J}} \cdot \boldsymbol{\theta}}{\hbar}}. \tag{7.3.10}$$

7.3.2 Rotations of Vector Operators

In order to further understand the properties of the angular momentum operator, let us consider some vector observable

$$\hat{\mathbf{V}} = (\hat{V}_x, \hat{V}_y, \hat{V}_z) \equiv (\hat{V}_1, \hat{V}_2, \hat{V}_3), \tag{7.3.11}$$

with three components such as, for example, the spin operator $\hat{\mathbf{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ or the position operator $\hat{\mathbf{r}} = (\hat{x}, \hat{y}, \hat{z})$. Intuitively, physical properties obtained rotating states or, alternatively, rotating observables should be the same. We then require that, for two arbitrary kets $|\psi\rangle$ and $|\phi\rangle$:

$$\langle\phi_{\boldsymbol{\theta}}|\hat{\mathbf{V}}|\psi_{\boldsymbol{\theta}}\rangle = \langle\phi|\hat{\mathbf{V}}_{\boldsymbol{\theta}}|\psi\rangle\,. \tag{7.3.12}$$

or, in words, that the matrix elements of the operators are the same regardless of whether we rotate the kets or we rotate the operator. In components, this reads

$$\langle\phi_{\boldsymbol{\theta}}|\hat{V}_i|\psi_{\boldsymbol{\theta}}\rangle = \sum_j R_{ij}(\boldsymbol{\theta})\,\langle\phi|\hat{V}_j|\psi\rangle\,, \tag{7.3.13}$$

and expanding the rotated states in terms of $\hat{D}(\boldsymbol{\theta})$, we have

$$\langle\phi|\hat{D}^\dagger(\boldsymbol{\theta})\hat{V}_i\hat{D}(\boldsymbol{\theta})|\psi\rangle = \sum_j R_{ij}(\boldsymbol{\theta})\,\langle\phi|\hat{V}_j|\psi\rangle\,, \tag{7.3.14}$$

and since this must be for all $|\psi\rangle$ and $|\phi\rangle$, this implies that

$$\hat{D}^\dagger(\boldsymbol{\theta})\hat{V}_i\hat{D}(\boldsymbol{\theta}) = \sum_j R_{ij}(\boldsymbol{\theta})\hat{V}_j. \tag{7.3.15}$$

If we now consider again the infinitesimal form of the rotation operator, we have that the left-hand side of the above reads

$$\hat{D}^\dagger(\boldsymbol{\theta})\hat{V}_i\hat{D}(\boldsymbol{\theta}) \simeq \Big(1+\frac{i}{\hbar}\sum_k \delta\theta_k\hat{J}_k\Big)\hat{V}_i\Big(1-\frac{i}{\hbar}\sum_k \delta\theta_k\hat{J}_k\Big) \tag{7.3.16}$$

$$= \hat{V}_i + \frac{i}{\hbar}\sum_k \delta\theta_k\Big[\hat{J}_k,\hat{V}_i\Big]. \tag{7.3.17}$$

In order to compute the right-hand side of Eq. (7.3.15), we recall that

$$\sum_j R_{ij}(\boldsymbol{\theta})\hat{V}_j \simeq \hat{V}_i + (\delta\boldsymbol{\theta}\times\hat{\mathbf{V}})\cdot\mathbf{e}_i \tag{7.3.18}$$

$$= \hat{V}_i + \Big(\sum_{i',j',k}\epsilon_{i'j'k}\delta\theta_{i'}\hat{V}_{j'}e_k\Big)\cdot\mathbf{e}_i \tag{7.3.19}$$

$$= \hat{V}_i + \sum_{i',j'}\epsilon_{i'j'k}\delta\theta_{i'}\hat{V}_{j'} \tag{7.3.20}$$

$$= \hat{V}_i - \sum_{k,j}\epsilon_{ijk}\delta\theta_k\hat{V}_j. \tag{7.3.21}$$

Equating the two, we then get

$$\frac{i}{\hbar}\Big[\hat{J}_k,\hat{V}_i\Big] = -\sum_j \epsilon_{ijk}\hat{V}_j, \tag{7.3.22}$$

$$\frac{i}{\hbar}\Big[\hat{J}_i,\hat{V}_k\Big] = \sum_j \epsilon_{ijk}\hat{V}_j, \tag{7.3.23}$$

$$\Big[\hat{J}_i,\hat{V}_j\Big] = -i\hbar\sum_j \epsilon_{ikj}\hat{V}_k, \tag{7.3.24}$$

and finally, exchanging $k \leftrightarrow j$ and removing the redundant sum, we recover a more familiar form:

$$\left[\hat{J}_i, \hat{V}_j\right] = i\hbar\epsilon_{ijk}\hat{V}_k. \tag{7.3.25}$$

We therefore see that the commutator relations among different components of the angular momentum are just a special case of the above when we take the vector operator to be the angular momentum itself: $\hat{\mathbf{V}} = \hat{\mathbf{J}}$.

7.4 General Properties of Angular Momentum Eigenstates

Having defined the characteristic commutation relations for the angular momentum operators, we are now ready to study some general properties of its eigenvalues and eigenvectors. It is useful to introduce another operator, defined as the sum of the squares of the different components of the angular momentum:

$$\hat{J}^2 = \hat{J}_x\hat{J}_x + \hat{J}_y\hat{J}_y + \hat{J}_z\hat{J}_z = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2\,. \tag{7.4.1}$$

This operator has the property that it commutes with all the components of the angular momentum.

Theorem $[\hat{J}^2, \hat{J}_\alpha] = 0$ *for any component of the angular momentum* $\alpha = (x, y, z)$.
Proof. Consider for example,

$$\begin{aligned}
[\hat{J}^2, \hat{J}_z] &= [\hat{J}_x\hat{J}_x + \hat{J}_y\hat{J}_y + \hat{J}_z\hat{J}_z, \hat{J}_z] = \\
&= [\hat{J}_x\hat{J}_x + \hat{J}_y\hat{J}_y, \hat{J}_z] = \\
&= \hat{J}_x[\hat{J}_x, \hat{J}_z] + [\hat{J}_x, \hat{J}_z]\hat{J}_x + \hat{J}_y[\hat{J}_y, \hat{J}_z] + [\hat{J}_y, \hat{J}_z]\hat{J}_y = \\
&= -i\hbar\hat{J}_x\hat{J}_y - i\hbar\hat{J}_y\hat{J}_x + i\hbar\hat{J}_y\hat{J}_x + i\hbar\hat{J}_x\hat{J}_y = 0\,.
\end{aligned}$$

The other cases are left as an exercise (see below).

Since $\hat{J}^2$ commutes with all the components of the angular momentum, it is possible to find simultaneous eigenkets of $\hat{J}^2$ and one of the three components. Mostly for historical reasons, it is the case that one chooses to find simultaneous eigenkets of $\hat{J}^2$ and $\hat{J}_z$, but other choices are possible. In the following, we will concentrate on the task of determining the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$.

Exercise 1.1 Show that the x and y components of the angular momentum operator commute with J^2, i.e.

$$\left[\hat{J}^2, \hat{J}_x\right] = 0, \quad \left[\hat{J}^2, \hat{J}_y\right] = 0\,. \tag{7.4.2}$$

7.4.1 Ladder Operators

Similarly to what already done when studying the harmonic oscillator, it is convenient to work here with non-Hermitian operators, called ladder operators, taken to be linear combinations of the x and y components of the angular momentum:

$$\hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y. \tag{7.4.3}$$

These operators do not commute, and satisfy

$$\begin{aligned}
[\hat{J}_+, \hat{J}_-] &= [\hat{J}_x + i\hat{J}_y, \hat{J}_x - i\hat{J}_y] && (7.4.4)\\
&= -i[\hat{J}_x, \hat{J}_y] + i[\hat{J}_y, \hat{J}_x] && (7.4.5)\\
&= 2i[\hat{J}_y, \hat{J}_x] && (7.4.6)\\
&= 2\hbar\hat{J}_z && (7.4.7)
\end{aligned}$$

and also

$$\begin{aligned}
[\hat{J}_z, \hat{J}_{\pm}] &= [\hat{J}_z, \hat{J}_x \pm i\hat{J}_y] && (7.4.8)\\
&= [\hat{J}_z, \hat{J}_x] \pm i[\hat{J}_z, \hat{J}_y] && (7.4.9)\\
&= i\hbar\hat{J}_y \pm i(-i\hbar\hat{J}_x) && (7.4.10)\\
&= \hbar(\pm\hat{J}_x + i\hat{J}_y) && (7.4.11)\\
&= \pm\hbar\hat{J}_{\pm}. && (7.4.12)
\end{aligned}$$

It is also easy to check that the ladder operators commute with $\hat{J}^2$, since they are just a linear combination of operators that commute with $\hat{J}^2$, thus

$$[\hat{J}^2, \hat{J}_{\pm}] = [\hat{J}^2, \hat{J}_x] \pm i[\hat{J}^2, \hat{J}_y] = 0. \tag{7.4.13}$$

7.4.2 Eigenvalues of $\hat{J}^2$ and $\hat{J}_z$

Armed with the ladder operators, we are now in position to derive the spectrum of eigenvalues of $\hat{J}^2$ and $\hat{J}_z$. Since we are looking for common eigenkets of these two operators, we write the eigenvalue equation as

$$\begin{aligned}
\hat{J}^2 |a, m\rangle &= \hbar^2 a |a, m\rangle , && (7.4.14)\\
\hat{J}_z |a, m\rangle &= \hbar m |a, m\rangle , && (7.4.15)
\end{aligned}$$

where the $\hbar$ factors are introduced for dimensionality consistency. We are thus labeling the common eigenkets with $|a, m\rangle$ and the real numbers a and m are the eigenvalues of the two operators and need to be determined from our analysis.
First of all, we analyze the action of the ladder operators on these eigenkets. For example, we can ask what is the ket resulting from the action of the ladder operators on the eigenkets. We first analyze the action of $\hat{J}_z$ on such states:

$$\begin{aligned}
\hat{J}_z\hat{J}_{\pm} |a, m\rangle &= \left([\hat{J}_z, \hat{J}_{\pm}] + \hat{J}_{\pm}\hat{J}_z\right) |a, m\rangle && (7.4.16)\\
&= \pm\hbar\hat{J}_{\pm} |a, m\rangle + \hbar m\hat{J}_{\pm} |a, m\rangle && (7.4.17)\\
&= \hbar(m \pm 1)\hat{J}_{\pm} |a, m\rangle , && (7.4.18)
\end{aligned}$$

thus we see that the ladder operators increase or decrease the eigenvalue of $\hat{J}_z$ by $\hbar$. They are called ladder operators precisely for this reason: they go up and down in the ladder of (discrete, as we will see) eigenvalues of the z component of the angular momentum. The behavior of $\hat{J}^2$ on the "laddered" states is different though:

$$\hat{J}^2 \hat{J}_\pm |a, m\rangle = \left([\hat{J}^2, \hat{J}_\pm] + \hat{J}_\pm \hat{J}^2 \right) |a, m\rangle \tag{7.4.19}$$

$$= \hbar^2 a \hat{J}_\pm |a, m\rangle \,, \tag{7.4.20}$$

thus the ladder operators *do not change* the eigenvalue associated to $\hat{J}^2$.
Another property we can readily identify is that

$$a \geq m^2, \tag{7.4.21}$$

as it can be demonstrated noticing that

$$\langle a, m | \hat{J}_x^2 | a, m\rangle = |\hat{J}_x |a, m\rangle |^2 \geq 0, \tag{7.4.22}$$

and similarly for the y component, thus

$$\hbar^2 a = \langle a, m | \hat{J}^2 | a, m\rangle \tag{7.4.23}$$

$$= \langle a, m | \hat{J}_x^2 | a, m\rangle + \langle a, m | \hat{J}_y^2 | a, m\rangle + \hbar^2 m^2 \tag{7.4.24}$$

$$\geq \hbar^2 m^2. \tag{7.4.25}$$

Because of this inequality, it follows that for fixed a, there must be a maximum allowed value of m, that we call m_{max}. Specifically, this is defined by the condition that the raising ladder operator cannot produce a state with larger eigenvalue $m_{\text{max}} + 1$, otherwise this would violate the inequality $a \geq m^2$. We thus have

$$\hat{J}_+ |a, m_{\text{max}}\rangle = 0. \tag{7.4.26}$$

Multiplying this equation from the left with $\hat{J}_-$ and using the fact that

$$\hat{J}_- \hat{J}_+ = (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y) \tag{7.4.27}$$

$$= \hat{J}_x^2 + \hat{J}_y^2 + i\hat{J}_x\hat{J}_y - i\hat{J}_y\hat{J}_x \tag{7.4.28}$$

$$= \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z, \tag{7.4.29}$$

we have

$$\hat{J}_- \hat{J}_+ |a, m_{\text{max}}\rangle = \left(\hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z \right) |a, m_{\text{max}}\rangle \tag{7.4.30}$$

$$= \hbar^2 \left(a - m_{\text{max}}^2 - m_{\text{max}} \right) |a, m_{\text{max}}\rangle = 0 \tag{7.4.31}$$

which is verified only if $(a - m_{\text{max}}^2 - m_{\text{max}}) = (a - m_{\text{max}}(1 + m_{\text{max}})) = 0$, thus for fixed a we have that the maximum allowed value of m is given by

$$m_{\text{max}}(1 + m_{\text{max}}) = a. \tag{7.4.32}$$

Using a similar argument, we can show that there must be also a minimum value m_{min}, thus

$$\hat{J}_+ |a, m_{\text{min}}\rangle = 0, \tag{7.4.33}$$

and in this case

$$\hat{J}_+\hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) \tag{7.4.34}$$

$$= \hat{J}_x^2 + \hat{J}_y^2 - i\hat{J}_x\hat{J}_y + i\hat{J}_y\hat{J}_x \tag{7.4.35}$$

$$= \hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z, \tag{7.4.36}$$

thus

$$\hat{J}_+\hat{J}_- |a, m_{\text{min}}\rangle = \left(\hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z\right) |a, m_{\text{min}}\rangle \tag{7.4.37}$$

$$= \hbar^2\left(a - m_{\text{min}}^2 + m_{\text{min}}\right) |a, m_{\text{min}}\rangle = 0, \tag{7.4.38}$$

yielding

$$m_{\text{min}}(m_{\text{min}} - 1) = a. \tag{7.4.39}$$

We therefore see that we must have $m_{\text{max}} = -m_{\text{min}}$ in order to satisfy both the equations above. Not only m takes quantized values, but it is bound in

$$-j \leq m \leq j, \tag{7.4.40}$$

where we have defined $j \equiv m_{\text{max}}$. Since the difference between the maximum and minimum eigenvalue must be an integer, we conclude that $2j$ = integer. Examples of possible values are

$$j = 0, \quad m = 0, \tag{7.4.41}$$

$$j = \frac{1}{2}, \quad m = -\frac{1}{2}, \frac{1}{2}, \tag{7.4.42}$$

$$j = 1, \quad m = -1, 0, 1, \tag{7.4.43}$$

$$j = \frac{3}{2} \quad m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \tag{7.4.44}$$

$$\ldots, \ldots. \tag{7.4.45}$$

Eigenstates of $\hat{J}^2$ are conventionally denoted by the integer or semi-integer j rather than a itself, and we have

$$\hat{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle, \tag{7.4.46}$$

$$\hat{J}_z |j, m\rangle = \hbar m |j, m\rangle. \tag{7.4.47}$$

7.4.3 Matrix Elements

We can now also determine other matrix elements for the angular momentum operators. Specifically, we have constructed the ket $|j,m\rangle$ in such a way that it is an eigen-ket of $\hat{J}^2$ and $\hat{J}_z$, thus the matrix representation of these operators will be diagonal. The main remaining matrix elements to be computed are between the other components of the angular momentum, $\hat{J}_x$ and $\hat{J}_y$. It is more convenient to work again with the ladder operators. We have seen that the $\hat{J}_+$ operator increases the m component by one, thus

$$\hat{J}_+ |j,m\rangle = C_+ |j,m+1\rangle\,, \tag{7.4.48}$$

where C_+ is a constant to be determined. We can determine C_+ considering that

$$|C_+|^2 = \left|\hat{J}_+ |j,m\rangle\right|^2 = \langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle\,, \tag{7.4.49}$$

and using the fact $\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z$, we get:

$$|C_+|^2 = \hbar^2[j(j+1) - m(m+1)]. \tag{7.4.50}$$

Similarly,

$$\hat{J}_- |j,m\rangle = C_- |j,m-1\rangle\,, \tag{7.4.51}$$

and

$$|C_-|^2 = \hbar^2[j(j+1) - m(m-1)]. \tag{7.4.52}$$

Thus, taking a convention where these matrix elements are real, we get

$$\hat{J}_+ |j,m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\;|j,m+1\rangle\,, \tag{7.4.53}$$

$$\hat{J}_- |j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\;|j,m-1\rangle\,. \tag{7.4.54}$$

7.5 The Spin-$1/2$ Case

In the previous sections, we have developed the general theory of angular momentum operators, satisfying the general commutation relation. As we have also discussed earlier, the concept of angular momentum is rather general and concerns the transformation of all 3-dimensional vector operators, not only real-space coordinates. In fact, we have already seen such an example, albeit in disguise. As we will now show, the spin operator is a specific example of a momentum operator.

7.5.1 The Representation

We recall that we have defined the spin 1/2 operator to be the 3-dimensional vector of operators:

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\Big(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\Big), \tag{7.5.1}$$

where we recall the definition of the Pauli matrices:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{7.5.2}$$

$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{7.5.3}$$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.5.4}$$

We can immediately identify the three components of the spin with those of a legitimate angular momentum operator: $(\hat{S}_x, \hat{S}_y, \hat{S}_z) \equiv (\hat{J}_x, \hat{J}_y, \hat{J}_z)$. We can in fact verify that

$$[\hat{S}_\alpha, \hat{S}_\beta] = i\hbar \hat{S}_\gamma \epsilon_{\alpha\beta\gamma}. \tag{7.5.5}$$

For example, explicit computation of this commutator in one case yields:

$$\frac{\hbar^2}{4}\Big(\hat{\sigma}_x\hat{\sigma}_y - \hat{\sigma}_y\hat{\sigma}_x\Big) = \frac{\hbar^2}{4}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right) = i\hbar\hat{S}_z. \tag{7.5.6}$$

The total angular momentum is

$$\hat{S}^2 = \frac{\hbar^2}{4}\Big(\hat{\sigma}_x^2 + \hat{\sigma}_y^2 + \hat{\sigma}_z^2\Big) = \frac{3}{4}\hbar^2\hat{I}, \tag{7.5.7}$$

where we have used the fact (easy to check) that each of the Pauli matrices squares to unity. We therefore see that in this specific case the spin operator corresponds to a momentum operator with $j = \frac{1}{2}$:

$$\hat{S}^2 \left|j = \frac{1}{2}, m = \pm\frac{1}{2}\right\rangle = \hbar^2\frac{3}{4}\left|j = \frac{1}{2}, m = \pm\frac{1}{2}\right\rangle, \tag{7.5.8}$$

$$\hat{S}_z \left|j = \frac{1}{2}, m = \pm\frac{1}{2}\right\rangle = \pm\frac{\hbar}{2}\left|j = \frac{1}{2}, m = \pm\frac{1}{2}\right\rangle. \tag{7.5.9}$$

Analogously, the ladder operators for spins 1/2, $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$, can also be explicitly computed:

$$\hat{S}_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{7.5.10}$$

$$\hat{S}_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{7.5.11}$$

and it is left as an exercise to show that these increase and decrease, respectively, the eigenvalues of $\hat{S}_z$.

7.5.2 Spins and Rotations

When we introduced the notion of spin, in the first lecture, we were already expecting the spin to be "quantum equivalent" of the intrinsic angular momentum of an object. In this sense, reconnecting the properties of the spin operator to those of the angular momentum is certainly reassuring. However, we have seen that one of the fundamental notions in quantum physics is that observables are connected to symmetry operations, and that the angular momentum is intrinsically connected to the notion of rotational operators. What kind of rotations then are connected to the spin degrees of freedom? To answer this question, we write down the rotation operator for the spin operator:

$$\hat{D}(\boldsymbol{\theta}) = e^{-\frac{i}{\hbar}\hat{\mathbf{S}}\cdot\boldsymbol{\theta}}. \tag{7.5.12}$$

To see that this is truly a rotation of the system, we can for example consider a rotation along the z direction, thus $\boldsymbol{\theta} = (0, 0, \theta_z)$ such that

$$\hat{D}(\theta_z) = e^{-\frac{i}{\hbar}\hat{S}_z\theta_z}. \tag{7.5.13}$$

We expect that the 3 components of the spin to rotate according to the rotation matrix along the z direction, which has the well-known form

$$\hat{R}(\theta_z) = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.5.14}$$

in such a way that

$$\langle \mathbf{S} \rangle_{\theta_z} = \hat{R}(\mathbf{S}). \tag{7.5.15}$$

We can easily verify that this equation is non-trivially satisfied by the spin operators. For example, if we take the expectation value of $\hat{S}_x$, after the rotation we have

$$\langle \hat{S}_x \rangle_{\theta_z} = \langle \Psi_{\theta_z} | \hat{S}_x | \Psi_{\theta_z} \rangle = \langle \Psi | e^{\frac{i}{\hbar}\hat{S}_z\theta_z} \hat{S}_x e^{-\frac{i}{\hbar}\hat{S}_z\theta_z} | \Psi \rangle, \tag{7.5.16}$$

where the form of the rotated operator is computed explicitly using the matrix representations:

$$e^{\frac{i}{\hbar}\hat{S}_z\theta_z} \hat{S}_x e^{-\frac{i}{\hbar}\hat{S}_z\theta_z} = \frac{\hbar}{2} \begin{pmatrix} e^{i\frac{\theta_z}{2}} & 0 \\ 0 & e^{-i\frac{\theta_z}{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\frac{\theta_z}{2}} & 0 \\ 0 & e^{i\frac{\theta_z}{2}} \end{pmatrix} \tag{7.5.17}$$

$$= \frac{\hbar}{2} \begin{pmatrix} e^{i\frac{\theta_z}{2}} & 0 \\ 0 & e^{-i\frac{\theta_z}{2}} \end{pmatrix} \begin{pmatrix} 0 & e^{-i\frac{\theta_z}{2}} \\ e^{i\frac{\theta_z}{2}} & 0 \end{pmatrix} \tag{7.5.18}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & e^{i\theta_z} \\ e^{-i\theta_z} & 0 \end{pmatrix} \tag{7.5.19}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & \cos\theta_z + i\sin\theta_z \\ \cos\theta_z - i\sin\theta_z & 0 \end{pmatrix} \tag{7.5.20}$$

$$= \hat{S}_x \cos\theta_z - \hat{S}_y \sin\theta_z. \tag{7.5.21}$$

Thus,

$$\langle \hat{S}_x \rangle_{\theta_z} = \langle \Psi | \hat{S}_x | \Psi \rangle \cos \theta_z - \langle \Psi | \hat{S}_y | \Psi \rangle \sin \theta_z, \tag{7.5.22}$$

$$\langle S_x \rangle_{\theta_z} = \langle S_x \rangle \cos \theta_z - \langle S_y \rangle \sin \theta_z, \tag{7.5.23}$$

as expected for the x component.

7.6 References and Further Reading

The discussion in this Chapter is adapted from Sakurai's *Modern Quantum Mechanics* (Chapter 3, sections 3.1, 3.2, 3.5), that presents a remarkably modern way of introducing angular momentum. Cohen-Tannoudji's book also contains a short discussion on the connection between rotations and Angular Momentum in its Chapter 6, at the beginning of complement B_{VI}, even though the main topic (Chapter 6, is not presented as in Sakurai's). As you might also see from other more "traditional" textbooks, one way of presenting angular momentum is by means of orbital angular momentum, a topic we won't introduce in this course before the next Chapter. As we will see, orbital angular momentum is just a special case of the angular momentum operators presented in this Chapter. Thus, the more fundamental (and maybe more elegant, depending on your taste) way of introducing angular momentum is not through orbital angular momentum but rather through rotation operators, as we have done in the previous discussion.